FOR THE NERDS:

From the Euler equation, $e^{ix} = \cos(x) + i\sin(x)$, so the set of complex numbers (or $\mathbb{C}$-line) is isomorphic to $\mathbb{R}^2$ (Cartesian product of 2 real lines). So to switch sin and cos neatly using $e^{ix} = \cos(x) + i\sin(x)$, you need to switch the real and imaginary components of these numbers, or switch the components of this 2-vector. This is a reflection in $\mathbb{R}^2$, and can be given by σ_x the x Pauli matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

All reflections and rotations in N^d can be characterised by the $O(N)$ orthogonal group, and so for the $\mathbb{C}$-line $\cong \mathbb{R}^2$, $O(2)$ will suffice. σ_x is naturally an element of $O(2)$, characterising a reflection in the y=x axis which is exactly what we would need. But to tie this back to the complex numbers now, the $U(1)$ unitary group characterises the subgroup of rotations in $O(2)$, but cannot describe reflections. But if you consider complex conjugation as a unary operator, this has the ability to map $z \to z^* \Leftrightarrow x+iy \to x-iy$, which can be viewed as a reflection in the x axis. So considering $U(1)$ with the unary map $C(z) : x+iy \to x-iy$, (which can be represented by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ which also happens to be σ_z the z Pauli matrix), we can make a group isomorphism to the $O(2)$ group by allowing this map to provide the concept of reflection.

The reason for this is, our vectors exist as elements of $U(1)$ which is an abelian group, as scalar multiplication is commutative. So the order of transformations do not matter. The same cannot be said for $O(2)$, as this doesn't hold for rotations combined with reflections. So we want to use the operations of the transformations from $O(2)$ (σ_x in particular) applied to elements of $U(1)$. These groups are fundamentally different, but complex conjugation operation (C) perfectly destroys the commutativity of the multiplication, to allow us to build an isomorphism.

We want to find the element of $U(1)$ that corresponds to σ_x in $O(2)$, which does exactly what we need by switching the imaginary and real axes. So a group isomorphism which inherently protects the underlying group structure, will allow us to understand what transformations to apply in $U(1)$, to flip the real and imaginary axes, to switch sine and cosine.

Diagonalising σ_x, then reversing gives you:

$$\sigma_x = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{H} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_z} \cdot \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{H^{-1}} = H\sigma_z H^{-1} = Ad(H)(\sigma_z)$$

Where H is the Hadamard gate as seen in quantum computing. If you notice, $H^2 = I = \sigma_x^2 \Leftrightarrow H = H^{-1}$ and $\sigma_x = \sigma_x^{-1}$. So:

$$\sigma_x^2 = I \Rightarrow \sigma_x^3 = \sigma_x = \sigma_x \cdot \underbrace{\big(H\sigma_z H^{-1}\big)}_{\sigma_x} \cdot \sigma_x^{-1} = \sigma_x = \big(\sigma_x H\big) \cdot \sigma_z \cdot \big(\sigma_x H\big)^{-1} \ \big[= Ad(\sigma_x H)(\sigma_z)\big].$$

Due to elements of $O(2)$ being able to represented by group of matrices $\hat{R}(x) = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$ With the hats being used henceforth to emphasise that the unary maps can be view as operators that act on a state and map you to another one.

We can immediately see applying σ_x to H corresponds to the $O(2)$ group representative of a rotation by $\frac{\pi}{4}$, hence the inverse of this matrix product by group isomorphism rotates by $-\frac{\pi}{4}$. Implicitly the isomorphism to $U(1)$ maps matrix multiplication to scalar multiplication, with the angle represented by the exponent. The underlying group structure is preserved allowing

$$\hat{\sigma}_x = \overbrace{\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{(\sigma_x H)}}^{\hat{R}\left(\frac{\pi}{4}\right)} \cdot \overbrace{\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_z}}^{\hat{C}} \cdot \overbrace{\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{(\sigma_x H)^{-1}}}^{\hat{R}\left(-\frac{\pi}{4}\right)} = \hat{R}\Big(\frac{\pi}{4}\Big) \circ \hat{C} \circ \hat{R}\Big(-\frac{\pi}{4}\Big)$$

In $U(1)$ group with complex conjugation operation, this allows σ_x to be represented as $\left(e^{i\frac{\pi}{4}} \circ \hat{C} \circ e^{-i\frac{\pi}{4}}\right)$, which we will call $\hat{\hat{\sigma}}_x$. Using $\hat{C}$ as conjugation map from earlier, this makes:

$$\hat{\hat{\sigma}}_x\big(e^{ix}\big) = \left(e^{i\frac{\pi}{4}} \circ \hat{C} \circ e^{-i\frac{\pi}{4}}\right)\big(e^{ix}\big) = e^{i\frac{\pi}{4}} \cdot \hat{C}\left(e^{i\left[x-\frac{\pi}{4}\right]}\right) = e^{i\frac{\pi}{4}} \cdot \left(e^{i\left[\frac{\pi}{4}-x\right]}\right) = e^{i\left[\frac{\pi}{2}-x\right]}$$

So $\hat{\hat{\sigma}}_x$ represents the mapping $e^{ix} \to e^{i\left[\frac{\pi}{2}-x\right]}$), and more abstractly $\hat{R}\big(x\big) \to \hat{R}\big(\frac{\pi}{2}-x\big)$ and so the transformation $x \to \frac{\pi}{2}-x$ turns sines into cosines and cosines into sines. $\square$

σ_x is also the NOT gate from quantum computing which interchanges states/components of $\mathbb{C}$ 2-vectors, so makes sense why σ_x plays a pivotal role here.

Summary:

To summarise, I treated $e^{ix} = \cos(x) + i\sin(x)$ as a 2-vector, to interchange sin and cos, I needed to find the how flip the vector components. Which I knew how to do using rotations and reflections – which is also the NOT gate in quantum computing.

Then constructed an underlying link between the complex numbers and these rotations and reflections by cleverly modifying the structure, while being careful not to break the connection, so I could translate all the information across to find out how to transform x.

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